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On the Structure of Reduced $\mathscr{T}$-Ternary Algebras of Degree Two

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INTRODUCTION

$\mathscr{T}$-Ternary algebras (see Section 1 for definition) arise naturally in a generalisation of Koecher's construction [9] of exceptional Lie algebras of type E_7 to a construction of arbitrary exceptional Lie algebras (over suitable fields) [1, 4]. The idea was to give models of Lie algebras in terms of special Jordan modules (instead of Jordan algebras as in [9]. Meanwhile, mainly under the influence of Kantor [8], a variation of the original notion of $\mathscr{T}$-ternary algebras came into discussion which is closely related to conservative algebras [7] and to a class of binary algebras with involution, called structurable algebras in [2]. This class of ternary algebras is somewhat more general over certain fields and it has the advantage that its connection with associative algebras and tensor products of composition algebras is somewhat more natural. On the other hand, the strong connection with Jordan modules is lost. It was shown in [5] that the ternary structure of reduced $\mathscr{T}$-ternary algebras is uniquely determined by its skew Hermitian $\mathscr{T}$-module structure, provided that the Jordan algebra in question is reduced of degree $\geqslant 2$. Paper [5] and the present one give a classification theory of reduced $\mathscr{T}$-ternary algebras using Jordan algebra theory. Although the main results are already known [2, 8], the method is essentially different from that used in [2, 8] and it might give more insight into the structure of these objects.

There is one striking difference between those $\mathscr{T}$-ternary algebras, whose Jordan algebra is of degree $\geqslant 3$ and those, whose Jordan algebra is of degree 2: In the first case, every symplectic special $\mathscr{T}$-module $\mathscr{V}$ admits a ternary composition such that $\mathscr{V}$ becomes a $\mathscr{T}$-ternary algebra, contrary to the degree 2 case. Thus $\mathscr{T}$-ternary algebras of degree 2 need a particular

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discussion. In [4] we have given (under certain restrictions) a necessary and sufficient condition for the dimension of those Clifford modules $\mathscr{V}$, which can be given a ternary composition such that $\mathscr{V}$ becomes a $\mathscr{J}$-ternary algebra of degree 2. In the present paper, we go another way, influenced by [8], namely, via a binary composition on the Jordan module under consideration, which gives a tensor product of two composition algebras as "coordinate algebra."

Section 1 gives a collection of previous results, which can be found in [5]. It also gives (Theorem 1.1) a generalisation of a proposition of [5] which is important for the proof of the existence theorem of regular elements. The importance of regular elements lies in the fact that they can be used to define a Peirce decomposition, which is the main tool for introducing the above-mentioned binary composition. Furthermore, Section 1 presents the examples which are shown to be the only simple reduced $\mathscr{J}$-ternary algebras of degree two.

Section 2 is an introduction for Section 3–5. It deals with so-called left neutral pairs, a generalisation of the notion of regular elements. Section 3 contains the study of the Peirce decomposition, Section 4 the imbedding of $\mathscr{J}(\mathscr{V})$ into $\mathscr{V}$, and Section 5 the definition and study of some properties of the binary product that leads to the "coordinate algebra" $\mathscr{C}_1 \otimes \mathscr{C}_2$, $\mathscr{C}_i$ a composition algebra. We study this product only so far as we need for the final section.

Section 6 contains the proof of the isomorphism theorem (Theorem 6.1), starting with a more detailed discussion of the (imbedded) Jordan algebra $\mathscr{J}(\mathscr{V})$ which leads to the definition of $\mathscr{C}_i$, and terminating with the explicit construction of an isomorphism in tye type (I) case (for type (II), see Proposition 6.2).

1. Preliminaries

Throughout we assume that K is a commutative field of characteristic not two or three. The notions module, vector space always mean unital K module or finite-dimensional vectors space over K, respectively. Linear (bi-, trilinear) mappings are supposed to be K linear in each argument.

Definition. A module $\mathscr{V}$ together with a trilinear mapping $(x, y, z) \mapsto xyz$ from $\mathscr{V} \times \mathscr{V} \times \mathscr{V}$ to $\mathscr{V}$ is said to be a $\mathscr{J}$-ternary algebra, if

(I) $xy(uvw) - uv(xyw) = (xyu)\,vw + u(yxv)\,w$,

(II) $xyz - zyx = zxy - xzy$

for all x, y, z, u, v, $w \in \mathscr{V}$.

The left and right multiplication operators are defined by

$$L(x,y)\colon\ \mathscr{V} \to \mathscr{V}, \qquad L(x,y)\,z := xyz,$$
$$R(x,y)\colon\ \mathscr{V} \to \mathscr{V}, \qquad R(x,y)\,z := zyx,$$

where $x, y, z \in \mathscr{V}$. We define a skew symmetric bilinear mapping

$$\langle -,- \rangle\colon\ \mathscr{V} \times \mathscr{V} \to \operatorname{lin}(\mathscr{V})$$

by

$$\langle x, y \rangle\, z := yzx - xzy. \tag{1.1}$$

By [5, Theorem 1.1] the linear span $\mathcal{J} = \mathcal{J}(\mathscr{V})$ of the set $\{\langle x,y \rangle;\ x,y \in \mathscr{V}\}$ in $\operatorname{lin}(\mathscr{V})$ is a Jordan subalgebra of $(\operatorname{lin}(\mathscr{V}))^+$, and $\langle -,- \rangle$ is a skew-Hermitian $\mathcal{J}$-form, i.e.,

$$\langle ax, y \rangle + \langle x, ay \rangle = 2a\langle x, y \rangle \tag{1.2}$$

for all $a \in \mathcal{J}$, x, $y \in \mathscr{X}$, where $a\langle x,y \rangle$ is the product of a and $\langle x,y \rangle$ in the Jordan algebra $\mathcal{J}$.

We have proved in [5, 1], that the following relations hold for all $a \in \mathcal{J}$ and $x, y, z, u, v \in \mathscr{V}$:

(III) $a(xyz) = (ax)\,yz - x(ay)\,z + xy(az)$,

(IV) $xyz - yxz = \langle x, y \rangle\, z$,

(V) $\langle xyu, v \rangle + \langle u, xyv \rangle = \langle x, \langle u, v \rangle\, y \rangle$,

(VI) $L(x, ay)\,z - L(y, ax)\,z = \langle x, y \rangle (az)$ $(z \in \mathscr{V})$.

Following [5, Sect. 4] a bilinear form β on the $\mathcal{J}$-ternary algebra $\mathscr{V}$ is called invariant, if

$$\beta(xyu, v) = -\beta(u, yxv) \tag{1.3}$$

for all $x, y, u, v \in \mathscr{V}$. By [5, Proposition 4.1], any skew-symmetric invariant bilinear form β satisfies

$$\beta(\langle x, y \rangle\, u, v) = \beta(u, \langle x, y \rangle\, v) = \beta(\langle u, v \rangle\, x, y), \tag{1.4}$$

$$\beta(xyu, v) = -\beta(x, vuy) \tag{1.5}$$

for all $x, y, u, v \in \mathscr{V}$. The most important example of a skew-symmetric invariant bilinear form is defined by

$$\beta_t(x, y) := \operatorname{trace}(\langle x, y \rangle) \qquad (x, y \in \mathscr{V}) \tag{1.6}$$

(see [5, Lemma 4.1]). Obviously, $\langle -, - \rangle$ is nondegenerate, if β_t is nondegenerate. On the other hand, if $\mathscr{T}(\mathscr{V})$ is nondegenerate (see [3, p. 39] for definition), then $\langle -, - \rangle$ nondegenerate implies β_t nondegenerate. Therefore [5, Theorem 6.1] can be formulated as follows: The $\mathscr{T}$-ternary algebra $\mathscr{V}$ is simple if and only if $\mathscr{T}(\mathscr{V})$ is simple and β_t is nondegenerate.

Let $c \neq e$ be an idempotent of $\mathscr{T} = \mathscr{T}(\mathscr{V})$, $\mathscr{V}$ an arbitrary $\mathscr{T}$-ternary algebra. Then by [5, Sect. 7] we have the Peirce decomposition of $\mathscr{V}$ with respect to c, i.e.,

$$\mathscr{V} = \mathscr{V}_1^c \oplus \mathscr{V}_0^c,$$

where

$$\mathscr{V}_i^c = \{x \in \mathscr{V}; cx = ix\}, \qquad i = 0, 1.$$

The following multiplication rules for the Peirce spaces $\mathscr{V}_1^c, \mathscr{V}_0^c$ are proved in [5, Proposition 7.1]:

$$\mathscr{V}_i^c \mathscr{V}_j^c \mathscr{V}_k^c \subset \mathscr{V}_{i-j+k}^c, \tag{1.7}$$

$$\langle \mathscr{V}_i^c, \mathscr{V}_j^c \rangle \subset \mathscr{T}_{1/2(i+j)}^c, \tag{1.8}$$

$$\mathscr{T}_i^c \mathscr{V}_j^c \subset \mathscr{V}_{2i-j}^c, \tag{1.9}$$

$$\beta(\mathscr{V}_1^c, \mathscr{V}_2^c) = \{0\}, \tag{1.10}$$

where β is any invariant bilinear form on $\mathscr{V}$. Moreover, by [5, Corollary to Proposition 7.1] we have

$$x_i y_i z_j = -\langle x_i, z_j \rangle y_i - \langle y_i, z_j \rangle x_i, \tag{1.11}$$

$$x_i y_j z_j = \langle x_i, y_j \rangle z_j \tag{1.12}$$

for all $x_i, y_i \in \mathscr{V}_i^c$; $y_j, z_j \in \mathscr{V}_j^c$; $i, j = 0, 1$; $i \neq j$.

PROPOSITION 1.1. *Let $c \neq e$ be an idempotent of $\mathscr{T} = \mathscr{T}(\mathscr{V})$, $\mathscr{V}$ a $\mathscr{T}$-ternary algebra over K, and $\mathscr{T} = \mathscr{T}_1^c \oplus \mathscr{T}_{1/2}^c \oplus \mathscr{T}_0^c$, $\mathscr{V} = \mathscr{V}_1^c \oplus \mathscr{V}_0^c$ the Peirce decompositions of $\mathscr{T}$ and $\mathscr{V}$, respectively. Then*

(a) $\langle x, xxx \rangle = 0$ *for all* $x \in \mathscr{V}_i^c$, $i = 0, 1$;

(b) *if* $\mathscr{T}_1^c = Kc$ *and* $\langle -, - \rangle$ *is nondegenerate on* $\mathscr{V}_1^c \times \mathscr{V}_1^c$, *then* $xxx = 0$ *for all* $x \in \mathscr{V}_1^c$. *If, moreover, K has at least five elements, then*

$$xyz = \tfrac{1}{2}(\langle x, y \rangle z - \langle y, z \rangle x + \langle z, x \rangle y)$$

for all $x, y, z \in \mathscr{V}_1^c$.

Proof. (a) An immediate consequence of (I) is $x^3xy + xx^3y = 0$ for all $x, y \in \mathscr{V}$. By (IV) we have $x^3xy - xx^3y = \langle x^3, x\rangle y$ for all $x, y \in \mathscr{V}$. Addition of these two equations yields

$$xx^3y = \tfrac{1}{2}\langle x, x^3\rangle y. \tag{1.13}$$

We suppose now that $x \in \mathscr{V}_i^c, y \in \mathscr{V}_j^c$ for $i \neq j$. By (1.12), (IV), and (1.13) we have $\langle y, x^3\rangle x = yx^3x = xx^3y + \langle x, y\rangle x^3 = \frac{1}{2}\langle x, x^3\rangle y + \langle x, y\rangle x^3$, hence

$$\langle y, x^3\rangle x + \langle y, x\rangle x^3 = \tfrac{1}{2}\langle x, x^3\rangle y. \tag{1.14}$$

Next we shall show that the left-hand side of (1.14) is zero. By (V) we have $\langle y, xxx\rangle = -\langle xxy, x\rangle + \langle x, \langle y, x\rangle x\rangle$. With $xxy = -2\langle x, y\rangle x$ (see (1.11)), we get

$$\langle y, x^3\rangle = 3\langle\langle x, y\rangle x, x\rangle. \tag{1.15}$$

We compute $\langle y, x\rangle x^3$ by means of (III), (1.8)–(1.11): $\langle x, y\rangle x^3 = (\langle x, y\rangle x)xx - x(\langle x, y\rangle x)x + xx(\langle x, y\rangle x) = (\langle x, y\rangle x)xx - xx(\langle x, y\rangle x) = \langle\langle x, y\rangle x, x\rangle x - 2\langle x, \langle x, y\rangle x\rangle x$. Hence

$$\langle x, y\rangle x^3 = 3\langle\langle x, y\rangle x, x\rangle x. \tag{1.16}$$

Combining (1.14)–(1.16), we have proved (a).

(b) Linearisation of $\langle x, x^3\rangle = 0$ yields $\langle y, x^3\rangle + \langle x, yxx\rangle + \langle x, xyx\rangle + \langle x, xxy\rangle = 0$ for all $x, y \in \mathscr{V}_1^c$. From this it follows by a repeated application of (V) that

$$4\langle y, x^3\rangle + 6\langle x, \langle x, y\rangle x\rangle = 0.$$

Since $\langle x, y\rangle \in \mathscr{T}_1^c = Kc$, $\langle x, y\rangle x$ is a scalar multiple of x, hence $\langle y, x^3\rangle = 0$ for all $x, y \in \mathscr{V}_1^c$. But now the assumption that $\langle -, -\rangle$ is nondegenerate on $\mathscr{V}_1^c \times \mathscr{V}_1^c$ forces x^3 to be zero and the first assertion of (b) has been proved. Again by linearisation, $x^3 = 0$ implies $xxy + xyx + yxx = 0$, hence by (IV) we get $3yxx + \langle x, y\rangle x + \langle y, x\rangle x = 0 = 3yxx$. Linearisation of $yxx = 0$ yields $yzx + yxz = 0$ for all $x, y, z \in \mathscr{V}_1^c$, which is, by (IV), equivalent to the assertion.

LEMMA 1.1. *If $\mathscr{V}$ is a nondegenerate $\mathscr{T}$-ternary algebra of degree two and $c \neq e$ an idempotent of $\mathscr{T}(\mathscr{V})$, then any $x = x_1 + x_0 \in \mathscr{V}$, $x_i \in \mathscr{V}_i^c$ satisfies*

(a) $x^3 = 3\langle x_1, x_0\rangle(x_1 - x_0)$,

(b) $L(x, x^3) = -3\langle x_1, x_0\rangle^2$.

Proof. Let $c \neq e$ be an idempotent of $\mathscr{T}(\mathscr{V})$ and $\mathscr{V} = \mathscr{V}_1^c \oplus \mathscr{V}_0^c$ the Peirce decomposition of $\mathscr{V}$ with respect to c (see Section 1 for definition). By (1.11) and (1.12) we have $x^3 = x_1^3 + x_0^3 - 2\langle x_1, x_0\rangle x_1 + \langle x_0, x_1\rangle x_1 - 2\langle x_0, x_1\rangle x_0 + \langle x_1, x_0\rangle x_0$ for all $x = x_1 + x_0 \in \mathscr{V}$, $x_i \in \mathscr{V}_i^c$. Since $x_i^3 = 0$ for all $x_i \in \mathscr{V}_i^c$, $i = 0, 1$ (Theorem 1.1), formula (a) has been proved. Now $\langle x_1, \langle x_1, x_0\rangle x_1\rangle = -\langle x_1, x_0 x_1 x_1\rangle = -\langle x_1, x_1 x_1 x_0\rangle - \langle x_1, \langle x_1, x_0\rangle x_1\rangle = \langle x_1^3, x_0\rangle - 2\langle x_1, \langle x_1, x_0\rangle x_1\rangle$. Thus $3\langle x_1, \langle x_1, x_0\rangle x_1\rangle = \langle x_1^3, x_0\rangle = 0$. The same computation shows that $\langle x_0, \langle x_1, x_0\rangle x_0\rangle = 0$. Furthermore, for all $y \in \mathscr{V}$ we have $x_1(\langle x_1, x_0\rangle x_1) y = x_1(\langle x_1, x_0\rangle x_1) y_0 = \langle x_1, \langle x_1, x_0\rangle x_1\rangle y_0 = 0$, hence $L(x_1, \langle x_1, x_0\rangle x_1) = 0$, and analogously $L(x_0, \langle x_1, x_0\rangle x_0) = 0$. Applying (a) we get $L(x, x^3) = -3L(x_1, \langle x_1, x_0\rangle x_0) + 3L(x_0, \langle x_1, x_0\rangle x_1) = -3\langle x_1, x_0\rangle\langle x_1, x_0\rangle$, the last equality by (VI), proving (b).

PROPOSITION 1.2. *If $\mathscr{V}$ is a nondegenerate reduced $\mathscr{T}$-ternary algebra of degree two, then there exists a mapping $q: \mathscr{V} \to K$, $q \not\equiv 0$, such that*

$$L(x, x^3) = q(x)\, Id$$

for all $x \in \mathscr{V}$.

Proof. We denote by μ the nondegenerate symmetric bilinear form satisfying $ab = \lambda(a)\, b + \lambda(b)\, a - \mu(a, b)\, e$, for all $a, b \in \mathscr{T} = \mathscr{T}(\mathscr{V})$, where $\lambda(a) = \mu(e, a)$, $a \in \mathscr{T}$. As usual we write $\mu(a) = \mu(a, a)$, $a \in \mathscr{T}$. Let $c \neq e$ be an idempotent of $\mathscr{T}$ and $\mathscr{V} = \mathscr{V}_1^c \oplus \mathscr{V}_0^c$ the corresponding Peirce decomposition of $\mathscr{V}$. We put

$$q_c(x) := -3\mu(\langle x_1, x_0\rangle)$$

for all $x = x_1 + x_0 \in \mathscr{V}$, $x_i \in \mathscr{V}_i^c$. Since $\langle x_1, x_0\rangle \in \mathscr{T}_{1/2}^c$ for all $x_i \in \mathscr{V}_i^c$ we have $\lambda(\langle x_1, x_0\rangle) = 0$ and thus $\langle x_1, x_0\rangle^2 = -\mu(\langle x_1, x_0\rangle)\, e$. By Lemma 1.1(b) we have $L(x, x^3) = q_c(x)\, Id$. Because μ is nondegenerate, there must exist an $a \in \mathscr{T}_{1/2}^c$ such that $a^2 \neq 0$. Since the elements $\langle x_1, x_0\rangle$, $x_i \in \mathscr{V}_i^c$ span $\mathscr{T}_{1/2}^c$, there exist $u_i \in \mathscr{V}_i^c$ with $\mu(\langle u_1, u_0\rangle) \neq 0$ which proves that q_c is not identically zero.

EXAMPLES. Let $\mathscr{C}_1, \mathscr{C}_2$ be composition algebras over the field K of characteristic $\neq 2, 3$ with standard involution τ_1 and τ_2, respectively. Then $\mathscr{X} = \mathscr{C}_1 \otimes \mathscr{C}_2$ has a canonical involution τ such that $\tau(x_1 \otimes x_2) = x_1^{\tau_1} \otimes x_2^{\tau_2}$ ($x_i \in \mathscr{C}_i$). Let $v_2 \in \mathscr{C}_2$ satisfy $v_2^{\tau_2} = -v_2$, $v_2^2 \neq 0$. Write $v = e_1 \otimes v_2$ (e_1 the unit element of $\mathscr{C}_1$).

For an arbitrary natural number $n \geqslant 1$ set

$$\mathscr{V} = \mathscr{X} \times \cdots \times \mathscr{X} \qquad (n \text{ times}).$$

Let $X=(x_1,\dots,x_n)$, $Y=(y_1,\dots,y_n)$, $Z=(z_1,\dots,z_n)\in\mathscr{V}$ and $x\in\mathscr{X}$. Put

$$x\cdot X=(xx_1, x^\tau x_2,\dots, x^\tau x_n)$$

and define $\varphi\colon\mathscr{V}\times\mathscr{V}\to\mathscr{X}$ by

$$\varphi(X,Y)=\sum_{i=1}^{n} x_i d_i y_i^\tau,$$

where d_1 is the unit element of $\mathscr{X}$ and $d_2,\dots,d_n$ are arbitrary elements of $\mathscr{X}$ with $d_i^\tau=d_i$. Define a trilinear composition on $\mathscr{V}$ by

$$\{XYZ\}=\varphi(X,v\cdot Y)\cdot Z+\varphi(Z,v\cdot Y)\cdot X-\varphi(Z,X)\cdot(v\cdot Y).$$

Then $\mathscr{V}$ is a $\mathscr{T}$-ternary algebra if either

(I_n) $\mathscr{C}_1,\mathscr{C}_2$ are both associative and n arbitrary, or

(II) at least one of $\mathscr{C}_1,\mathscr{C}_2$ is not associative and $n=1$.

In either case $\mathscr{V}$ is reduced and simple and of degree two.

2. Left Neutral Pairs

Definition. A pair (u,v) of elements of a $\mathscr{T}$-ternary algebra $\mathscr{V}$ is called *left neutral*, if

$$L(u,v)=Id.$$

Lemma 2.1. *If (u,v) is left neutral, then*

$$R(u,v)^2-4R(u,v)+3Id=0.$$

Proof. This follows directly from (V) by replacing $x\to u$, $y\to v$, $u\to x$ for an arbitrary $x\in\mathscr{V}$, and $v\to u$.

So we get a direct vector space sum

$$\mathscr{V}=\mathscr{V}_1(u,v)\oplus\mathscr{V}_3(u,v),$$

where

$$\mathscr{V}_i=\{x\in\mathscr{V};\,xvu=ix\},\qquad i=1,3.$$

Lemma 2.2.

$$\mathscr{V}_1(u,v)=\{x\in\mathscr{V};\,\langle u,x\rangle=0\},\qquad \mathscr{V}_3(u,v)=\{x\in\mathscr{V};\,\langle u,x\rangle v=2x\}.$$

Proof. $x \in \mathscr{V}_1(u,v) \Leftrightarrow xvu = x \Leftrightarrow xvu = uvx \Leftrightarrow \langle u, x\rangle v = 0 \Rightarrow \langle u, x\rangle = \langle uvu, x\rangle = -\langle u, uvx\rangle + \langle u, \langle u, x\rangle v\rangle = -\langle u, x\rangle. \Rightarrow 2\langle u, x\rangle = 0$. Since $\operatorname{char} K \neq 2$ we have $\langle u, x\rangle = 0$. The other inclusion as well as the assertion for $\mathscr{V}_3(u,v)$ is obvious.

Let (u, v) be a fixed left neutral pair. We write

$$r(x) := xvu, \qquad \varphi(x) := vxv, \qquad \psi(x) := uxu$$

for all $x \in \mathscr{V}$. From (I) we get $u(vxv)u = -(xvu)vu$, hence

$$\psi \circ \varphi = -r^2. \tag{2.1}$$

Since r is bijective (Lemma 2.1), φ is injective and ψ is surjective, so φ and ψ are bijective for we assume $\dim \mathscr{V} < \infty$ throughout this article. We set

$$\tilde{x} := s(x) := 2r^{-1}(x) - x \qquad (x \in \mathscr{V}),$$

and

$$\hat{x} := -(\varphi \circ s)(x) \qquad (x \in \mathscr{V}).$$

These are bijective linear mappings.

PROPOSITION 2.1. *For all $x, y, z \in \mathscr{V}$ we have*

$$(x\hat{y}z)\hat{} = -\hat{x}y\hat{z}.$$

Proof. First we show how this follows from the special case

$$(xvy)\hat{} = -\hat{x}u\hat{y} \qquad (x, y \in \mathscr{V}) \tag{2.2}$$

by means of (I): Since

$$xy^{\varphi}z = yv(xvz) - xv(yvz) - (yvx)vz$$

for all $x, y, z \in \mathscr{V}$, (2.2) yields $(xy^{\varphi}z)\hat{} = -\hat{y}u(\hat{x}u\hat{z}) - \hat{x}u(\hat{y}u\hat{z}) - (\hat{y}u\hat{x})u\hat{z} = \hat{x}\hat{y}^{\psi}\hat{z} = -\hat{x}((\psi \circ \varphi \circ s)(y))\hat{z} = \hat{x}((r^2 \circ s)(y))\hat{z}$, hence $(x\hat{y}z)\hat{} = -\hat{x}((r^2 \circ s^2)(y))\hat{z}$. With the notation

$$x^{\tau} := 2x - x^{r} \qquad (x \in \mathscr{V}),$$

an easy computation gives

$$\tau = -r \circ s = -s \circ r, \qquad \tau^2 = Id,$$

and this yields $(x\hat{y}z)\hat{} = -\hat{x}y\hat{z}$. To prove (2.2) we put

$$x \cdot y := xvy \qquad (x, y \in \mathscr{V})$$

and write $x * y := \varphi^{-1}(\varphi(x)\, u\varphi(y))$. Then, $(2.2) \Leftrightarrow s(x \cdot y) = s(x) * s(y) \Leftrightarrow (r \circ s)(x^r * y^r) = r(x^{rs} * y^{rs}) \Leftrightarrow \tau(x^r * y^r) = r(x^\tau * y^\tau) \Leftrightarrow (r \circ \tau)(x^r * y^r) = r^2(x^\tau * y^\tau) \Leftrightarrow (2\tau - Id)((2x - x^\tau)(2y - y^\tau)) = -(x^{\tau\varphi} * y^{\tau\varphi})^{\varphi}$. $(*)$

An immediate concequence of (V) is

$$(x \cdot y)^r = x \cdot y + [x, y]. + [x^r, y]. - [x, y^r].,\tag{2.3}$$

where $[x, y]. := x \cdot y - y \cdot x$, and this gives

$$(x \cdot y)^\tau = y \cdot x + [x^\tau, y]. - [x, y^\tau]..\tag{2.4}$$

Using (I), a bit lengthy but simple computation shows that Eq. $(*)$ follows from (2.4), which proves (2.2).

COROLLARY. *If (u, v) is left neutral, then $(-v, u)$ is left neutral too.*

Proof. Replacing x by u in (2.2) gives $\hat{y} = (uvy)\hat{} = -\hat{u}u\hat{y}$, hence $-(vu^s v)\, ux = -x$ for all $x \in \mathscr{V}$. Since $u^s = u$, it follows from (I) that the left-hand side is $v(uvu)\, x = vux$, which forces $L(v, u) = -Id$.

Remark 1. For later use we note two immediate consequences of (I), namely,

$$\hat{x}uy = vx^\tau y \qquad (x, y \in \mathscr{V}),\tag{2.5}$$

$$x^\tau vy = u\hat{x}y \qquad (x, y \in \mathscr{V}).\tag{2.6}$$

Remark 2. Write $\{xyz\} = x\hat{y}z$ for $x, y, z \in \mathscr{V}$. Then Proposition 2.1 implies that (I) is equivalent to

$$\{xy\{abc\}\} - \{ab\{xyc\}\} = \{\{xya\}\, bc\} - \{a\{yxb\}\, c\}\tag{2.7}$$

for x, y, a, b, $c \in \mathscr{V}$. This together with

$$\{xyz\} = x \cdot (y^s \cdot z) - y^s \cdot (x \cdot z) + (y^s \cdot x) \cdot z\tag{2.8}$$

can be viewed as an identity for the algebra $(\mathscr{V}, \cdot)$ involving s. Proposition 2.2 shows that this identity of degree five is equivalent to one of degree four.

PROPOSITION 2.2. *Assume that there exists a left neutral pair $(u, v) \in \mathscr{V} \times \mathscr{V}$. Then with $x \cdot y = xvy (x, y \in \mathscr{V})$, (I) is equivalent to*

$$w \cdot (x\hat{y}z) = (w \cdot x)\, \hat{y}z - x(w^\tau \cdot y)\hat{}\, z + x\hat{y}(w \cdot z)\tag{2.9}$$

for x, y, z, $w \in \mathscr{V}$.

Proof. We write (2.7) and (2.8) in operator form. Set $L'(x,y)z = x\hat{y}z$, $L(x)y = x\cdot y = xvy$ for $x,y,z\in\mathscr{V}$. Then (2.6), (2.8), and (2.9) respectively reads as

$$L(x) = L'(u, x^\tau), \tag{2.10}$$

$$L'(x,y) = [L(x), L(y^s)] + L(y^s\cdot x), \tag{2.11}$$

$$[L(w), L'(x,y)] = L'(w\cdot x, y) - L'(x, w^\tau\cdot y). \tag{2.12}$$

Because (I) is equivalent to (2.7), we have to show

$$[L'(x,y), L'(a,b)] = L'(L'(x,y)a, b) - L'(a, L'(y,x)b).$$

Now,

$$\begin{aligned}
[L'(x,y), L'(a,b)] &= [[L(x), L(y^s)], L'(a,b)] \\
&\quad + [L(y^s\cdot x), L'(a,b)] = [[L(x), L'(a,b)], L(y^s)] \\
&\quad + [L(x), [L(y^s), L'(a,b)]] + [L(y^s\cdot x), L'(a,b)] \\
&= [L'(x\cdot a, b), L(y^s)] - [L'(a, x^\tau\cdot b), L(y^s)] \\
&\quad + [L(x), L'(y^s\cdot a, b)] - [L(x), L'(a, y^{s\tau}\cdot b)] \\
&\quad + [L(y^s\cdot x), L'(a,b)] = -L'(y^s\cdot(x\cdot a), b) \\
&\quad + L'(x\cdot a, v^{s\tau}\cdot b) + L'(y^s\cdot a, x^\tau\cdot b) - L'(a, y^{s\tau}\cdot(x^\tau\cdot b)) \\
&\quad + L'(x\cdot(y^s\cdot a), b) - L'(y^s\cdot a, x^\tau\cdot b) - L'(x\cdot a, y^{s\tau}\cdot b) \\
&\quad + L'(a, x^\tau\cdot(y^{s\tau}\cdot b)) + L'((y^s\cdot x)a, b) - L'(a, (y^s\cdot x)^\tau\cdot b) \\
&= L'(L'(x,y)a, b) - L'(a, y^{s\tau}\cdot(x^\tau\cdot b) - x^\tau\cdot(y^{s\tau}\cdot b) \\
&\quad + (y^s\cdot x)^\tau\cdot b).
\end{aligned}$$

It remains to show that

$$L'(y,x)b = y^{s\tau}\cdot(x^\tau\cdot b) - x^\tau\cdot(y^{s\tau}\cdot b) + (y^s\cdot x)^\tau\cdot b.$$

By (2.10) and (2.3) the right-hand side is equal to $[L(y^{s\tau}), L'(u,x)]b + L'(u, y^s\cdot x)b$ and this equals $L'(y^{s\tau}\cdot u, x)b - L'(u, y^s\cdot x)b + L'(u, y^s\cdot x)b$ because of (2.12). But $s\circ\tau\circ r = \tau^2 = Id$ which proves the proposition.

3. Imbedding of $\mathcal{T}(\mathcal{V})$ into $\mathcal{V}$

PROPOSITION 3.1. *If (u, v) is left neutral in $\mathcal{V} \times \mathcal{V}$, then $\mathcal{V}_3(u, v)$ is a Jordan algebra with respect to the composition*

$$x \circ y := \tfrac{1}{4}(\langle u, x\rangle y + \langle u, y\rangle x),$$

and

$$\eta\colon \mathcal{T}(\mathcal{V}) \to \mathcal{V}_3(u, v), \qquad \eta(a) := av$$

is an isomorphism from $\mathcal{T}(\mathcal{V})$ onto $(\mathcal{V}_3(u, v), \circ)$ with inverse

$$\eta^{-1}(x) = \tfrac{1}{2}\langle u, x\rangle \qquad (x \in \mathcal{V}_3(u, v)).$$

Proof. First we show

$$\langle u, av\rangle = 2a \qquad (a \in \mathcal{T}(\mathcal{V})).$$

Since $\mathcal{T}(\mathcal{V})$ is the linear span of the endomorphisms $\langle x, y\rangle$, $x, y \in \mathcal{V}$, we may assume that $a = \langle x, y\rangle$ for suitable $x, y \in \mathcal{V}$. Then $\langle u, av\rangle = \langle u, yvx\rangle - \langle u, xvy\rangle = -\langle yvu, x\rangle + \langle y, \langle u, x\rangle v\rangle + \langle xvu, y\rangle - \langle x, \langle u, y\rangle v\rangle = -\langle uvy, x\rangle - \langle\langle u, y\rangle v, x\rangle + \langle y, \langle u, x\rangle v\rangle + \langle uvx, y\rangle + \langle\langle u, x\rangle v, y\rangle - \langle x, \langle u, y\rangle v\rangle = 2\langle x, y\rangle = 2a$. In particular Lemma (2.2) yields $\eta(\mathcal{T}(\mathcal{V})) \subset \mathcal{V}_3(u, v)$. We write $\theta := \frac{1}{2}\langle u, -\rangle$ and find $\theta \circ \eta(a) = \frac{1}{2}\langle u, av\rangle = a$ and $\eta \circ \theta(x) = \frac{1}{2}\langle u, x\rangle v = x$ (Lemma 2.2). Hence η is bijective with $\eta^{-1} = \theta = \frac{1}{2}\langle u, -\rangle$.

We take $a, b \in \mathcal{T}(\mathcal{V})$ and can assume that $a = \langle u, x\rangle$, $b = \langle u, y\rangle$ for suitable $x, y \in \mathcal{V}_3(u, v)$, for we have just shown that

$$\mathcal{T}(\mathcal{V}) = \langle u, \mathcal{V}_3(u, v)\rangle. \tag{3.1}$$

We conclude that $\eta(a \cdot b) = \eta(\langle u, x\rangle\langle u, y\rangle) = \frac{1}{2}\langle u, x\rangle(\langle u, y\rangle v) + \frac{1}{2}\langle u, y\rangle(\langle u, x\rangle v) = \langle u, x\rangle y + \langle u, y\rangle x = 4x \circ y$. On the other hand, $\eta(a) \circ \eta(b) = (av) \circ (bv) = (\langle u, x\rangle v) \circ (\langle u, y\rangle v) = 4x \circ y$.

We consider the case where $\mathcal{V}$ is a $\mathcal{T}$-ternary algebra of degree two. That means that there is a linear form λ and a symmetric bilinear form μ on $\mathcal{T} = \mathcal{T}(\mathcal{V})$ such that

$$ab = \lambda(a)\, b + \lambda(b)\, a - \mu(a, b)\, e \tag{3.2}$$

for all $a, b \in \mathcal{T}$, where e is the unit element of $\mathcal{T}$. Putting $a = \eta^{-1}(x), b = \eta^{-1}(y)$ and applying η we get

$$x \circ y = \lambda(\eta^{-1}(x))\, y + \lambda(\eta^{-1}(y))\, x - \mu(\eta^{-1}(x), \eta^{-1}(y))\, v.$$

We set

$$t(x) := \lambda(\eta^{-1}(x)) \qquad \text{for} \quad x \in \mathscr{V}_3(u, v)$$

and

$$(x, y) := \mu(\eta^{-1}(x), \eta^{-1}(y)) \qquad \text{for} \quad x, y \in \mathscr{V}_3(u, v).$$

This yields

$$x \circ y = t(x)\, y + t(y)\, x - (x, y)\, v \tag{3.3}$$

for all $x, y \in \mathscr{V}_3(u, v)$. Note that x is invertible in $(\mathscr{V}_3(u, v), \circ)$ if and only if $(x, x) \neq 0$. Furthermore, $x \neq v$ is an idempotent in $(\mathscr{V}_3(u, v), \circ)$ if and only

$$t(x) = \tfrac{1}{2} \qquad \text{and} \qquad (x, x) = 0. \tag{3.4}$$

From what we have mentioned in Section 1, concerning invariant bilinear forms on $\mathscr{T}$-ternary algebras, it is clear that

$$\beta(x, y) := \tfrac{1}{2}\lambda(\langle x, y\rangle) \tag{3.5}$$

defines a skew-symmetric invariant bilinear form on $\mathscr{V}$. Moreover, $t(x) = \lambda(\frac{1}{2}\langle u, x\rangle)$, and so

$$t(x) = \beta(u, x) \qquad (x \in \mathscr{V}_3(u, v)). \tag{3.6}$$

Since $\mu(a, b) = 2\lambda(a)\, \lambda(b) - \lambda(ab)$, we get

$$(x, y) = 2\beta(u, x)\, \beta(u, y) - \tfrac{1}{2}\beta(\langle u, x\rangle\, u, y) \qquad (x, y \in \mathscr{V}_3(u, v)). \tag{3.7}$$

4. Regular Elements and the Peirce Decomposition

Definition. An element u in a $\mathscr{T}$-ternary algebra is called regular, if there exists $\kappa \in K$, $\kappa \neq 0$ such that

$$L(u, uuu) = \kappa Id.$$

Proposition 4.1. *Let $\mathscr{V}$ be a nondegenerate reduced $\mathscr{T}$-ternary algebra of degree two. Then $\mathscr{V}$ contains regular elements. Moreover, every nondegenerate $\mathscr{T}$-ternary subalgebra $\mathscr{U}$ of $\mathscr{V}$ contains regular elements of $\mathscr{V}$.*

Proof. The first assertion is an immediate consequence of Proposition 1.2. This result applied to $\mathscr{U}$ (which is necessarily of degree two) shows that $\mathscr{U}$ contains a regular element (of $\mathscr{U}$), say, w. Hence there exists

$\chi \in K$, $\chi \neq 0$, such that $w(www)y = \chi y$ for all $y \in \mathscr{U}$. On the other hand, by proposition 1.2 there exists $\kappa \in K$, satisfying $w(www)x = \kappa x$ for $x \in \mathscr{V}$, hence $\chi = \kappa \neq 0$, proving that w is regular in $\mathscr{V}$.

From now on we denote by u an arbitrary but fixed regular element and $\kappa \neq 0$ a corresponding scalar satisfying $L(u, u^3) = \kappa Id$. We write

$$v := \frac{1}{\kappa} u^3. \tag{4.1}$$

Hence (u, v) is left neutral and we can apply the results of Section 2 on left neutral pairs, particularly

$$L(v, u) = -Id = -L(u, v). \tag{4.2}$$

Moreover,

$$L(u, u) = \tfrac{1}{3}\kappa L(v, v), \tag{4.3}$$

because (I) implies $u^3vx = -u(uuv)x$ for all $x \in \mathscr{V}$, and since $uuv = vuu + \langle v, u\rangle u = -u - 2u$, we get $\kappa vvu = 3uux$. Formula (4.3) yields $-3u = uuv = \frac{1}{3}\kappa v^3$, hence

$$v^3 = -9\kappa^{-1}u. \tag{4.4}$$

By the results of Section 2 we have decompositions

$$\mathscr{V} = \mathscr{V}_1(u, v) \oplus \mathscr{V}_3(u, v)$$

and

$$\mathscr{V} = \mathscr{V}_1(-v, u) \oplus \mathscr{V}_3(-v, u),$$

where

$$\mathscr{V}_i(u, v) = \{x \in \mathscr{V}; xvu = ix\},$$
$$\mathscr{V}_i(-v, u) = \{x \in \mathscr{V}; xuv = -ix\}.$$

In the present situation where we assume that u is a fixed regular element, we write for the sake of brevity

$$\mathscr{V}_i := \mathscr{V}_i(u, v), \qquad \mathscr{V}_{-i} := \mathscr{V}_i(-v, u), \qquad i = 1, 3.$$

We claim that $\mathscr{V}$ is the direct vector space sum of the subspaces $\mathscr{V}_i \cap \mathscr{V}_{-j}$ with $i, j = 1, 3$.

LEMMA 4.1. *$L(u,u)$ maps $\mathscr{V}_i$ into $\mathscr{V}_{-i}$ for $i=\pm 1, \pm 3$.*

Proof. An immediate consequence of (IV) is

$$xxy - 2yxx + xyx = 0 \qquad (x, y \in \mathscr{V}). \tag{4.5}$$

Now $x \in \mathscr{V}_{-1} \Leftrightarrow xuv = -x \Rightarrow (uxu)vu = -u(xuv)u = uxu \Rightarrow uxu \in \mathscr{V}_{-1}$ $\Rightarrow \langle u, uxu\rangle = 0$. Thus by (4.5) $\langle u, uux\rangle = 2\langle u, xuu\rangle$. On the other hand, $\langle u, uux\rangle = -\langle u^3, x\rangle + \langle u, \langle u, x\rangle u\rangle = -\kappa\langle v, x\rangle + \langle u, uxu\rangle - \langle u, xuu\rangle = -\langle u, xuu\rangle$. We conclude

$$\langle u, uux\rangle = \langle u, uxu\rangle = \langle u, xuu\rangle = 0 \qquad (x \in \mathscr{V}_{-1}). \tag{4.6}$$

Hence $L(u,u)x \in \mathscr{V}_1$ for all $x \in \mathscr{V}_{-1}$. The same calculation with u replaced by v and v replaced by u gives

$$\langle v, vvx\rangle = \langle v, vxv\rangle = \langle v, xvv\rangle = 0 \qquad (x \in \mathscr{V}_1). \tag{4.7}$$

This and (4.3) imply $L(u,u)x \in \mathscr{V}_{-1}$ for all $x \in \mathscr{V}_1$.

Now $x \in \mathscr{V}_{-3} \Rightarrow xuv = -2x \Rightarrow (uxu)vu = -u(xuv)u = 2uxu \Rightarrow uxu \in \mathscr{V}_3 \Rightarrow \langle u, uxu\rangle v = 2uxu$. Hence by (4.5) we have

$$\langle u, uux\rangle v - 2\langle u, xuu\rangle v = -2uxu \qquad (x \in \mathscr{V}_{-3}). \tag{4.8}$$

On the other hand, $\langle u, uux\rangle = -\langle u^3, x\rangle + \langle u, \langle u, x\rangle u\rangle = -\kappa\langle v, x\rangle + \langle u, uxu\rangle - \langle u, xuu\rangle$. This combined with (4.8) yields

$$3\langle u, uux\rangle v = 2uxu - 2\kappa\langle v, x\rangle v \qquad (x \in \mathscr{V}_{-3}). \tag{4.9}$$

The right-hand side is equal to $2uxu - 2\langle v, x\rangle u^3 = 2uxu + 4xuu - 4\,uxu + 4uux = 4uux + (4xuu - 2uxu) = 4uux + 2uux$, the latter equality because of (4.8). Hence

$$\langle u, uux\rangle v = 2uux \qquad (x \in \mathscr{V}_{-3}), \tag{4.10}$$

in other words, $L(u,u)x \in \mathscr{V}_3$ for all $x \in \mathscr{V}_{-3}$. Along the same line one verifies

$$\langle v, vvx\rangle u = -2vvx \qquad (x \in \mathscr{V}_3) \tag{4.11}$$

and the lemma has been proved.

We let

$$\mathscr{V}_{ij} := \mathscr{V}_i \cap \mathscr{V}_{-j}, \qquad i,j = 1, 3.$$

Note that

$$u \in \mathscr{V}_{13}, \qquad v \in \mathscr{V}_{31}.$$

PROPOSITION 4.2. *$R(u, v)$ and $R(v, u)$ commute, in other words, we have the direct vector space sum*

$$\mathscr{V} = \mathscr{V}_{11} \oplus \mathscr{V}_{13} \oplus \mathscr{V}_{31} \oplus \mathscr{V}_{33}.$$

Proof. We show that $\mathscr{V}_{-i}$ is invariant under $R(u, v)$ for $i = 1, 3$. First let $x \in \mathscr{V}_{-1}$. That means $xuv = -x$ or equivalently $\langle x, v\rangle = 0$. Hence $\langle xvu, v\rangle = -\langle u, xvv\rangle + \langle x, \langle u, v\rangle v\rangle = -\langle u, vvx\rangle - \langle u, \langle v, x\rangle v\rangle + 2\langle x, v\rangle = -\langle u, vvx\rangle$. The last expression is zero in view of (4.3) and Lemma 4.2, hence $\langle xvu, v\rangle = 0$ and this is equivalent to $xvu \in \mathscr{V}_{-1}$. To show that $\mathscr{V}_{-3}$ is invariant under $R(u, v)$, we prove first

$$9\langle u, x\rangle u = \kappa(4vxv - 6xvv) \quad (x \in \mathscr{V}_3), \qquad uxu = \kappa xvv \quad (x \in \mathscr{V}_{-3}). \quad (4.12)$$

$$\begin{aligned} x \in \mathscr{V}_{-3} \Rightarrow \langle x, v\rangle u = 2x \Rightarrow -3vvx + 3xvv &= -3\langle x, v\rangle v \\ &= -3\kappa^{-1}\langle x, v\rangle u^3 = -3\kappa^{-1}(2xuu - 2uxu + 2uux) \\ &= -3\kappa^{-1}((2xuu - uxu - uux) - uxu + 3uux) \\ &= 3\kappa^{-1}(-uxu + 3uux) = 3\kappa^{-1}uxu - 3vvx. \end{aligned}$$

$$\begin{aligned} x \in \mathscr{V}_3 \Rightarrow -9\kappa^{-1}\langle u, x\rangle u = \langle u, x\rangle v^3 &= 2xvv - 2vxv + 2vvx \\ &= (2vvx - 4xvv + 2vxv) + 6xvv - 4vxv \\ &= 6xvv - 4vxv, \end{aligned}$$

and this finishes the proof of (4.12).

For all $x \in \mathscr{V}_{-3}$ we have $uxu \in \mathscr{V}_3$ (this is immediate by (I)), hence by (4.12) $9\kappa\langle xvu, v\rangle u = 9\kappa(-\langle u, xvv\rangle u + \langle x, \langle u, v\rangle v\rangle u) = -9\langle u, uxu\rangle u + 36\kappa x = \kappa((uxu)vv - 4v(uxu)v + 36x) \Rightarrow 9\langle xvu, v\rangle u = -6u(xuv)v + 4(xuv)uv + 36x = 18uxv + 36x + 36x = 18xuv + 18\langle u, x\rangle v + 72x = -54x + 18xvu - 18x + 72x = 18xvu$, hence $\langle xvu, v\rangle u = 2xvu$ and this forces xvu to be in $\mathscr{V}_{-3}$ again.

We shall have several opportunities to make use of the following formulas, which can easily be deduced from (4.3), (4.5), and (4.12):

$$\begin{gathered} vvx = 3xvv = -3vxv \quad (x \in \mathscr{V}_{13}), \qquad uux = 3xuu = -3uxu \quad (x \in \mathscr{V}_{31}), \\ 3vvx = -3xvv = -vxv, \qquad 3uux = -3xuu = -uxu \quad (x \in \mathscr{V}_{33}). \end{gathered} \quad (4.13)$$

In Section 2 we have defined $\hat{x} = -vx^s v$ for $x \in \mathscr{V}$, where $s = 2r^{-1} - Id$. Since $x^s = -\frac{1}{3}x$ for $x \in \mathscr{V}_3$ and $x^s = x$ for $x \in \mathscr{V}_1$, it is easily seen that (4.13) implies

$$\begin{aligned} \hat{x} &= +\tfrac{1}{3}vvx, \qquad \text{if} \quad x \in \mathscr{V}_{13} \oplus \mathscr{V}_{31}, \\ &= -vvx, \qquad \text{if} \quad x \in \mathscr{V}_{11} \oplus \mathscr{V}_{33}. \end{aligned} \quad (4.14)$$

Proposition 4.3. *The derivation $L(u,u)$ maps $\mathscr{V}_{ij}$ bijectively onto $\mathscr{V}_{ji}$ for $i,j=\pm 1, \pm 3$ and satisfies*

$$\begin{aligned} L(u,u)^2 x &= -3\kappa x, && \text{for} \quad x \in \mathscr{V}_{13} \oplus \mathscr{V}_{31}, \\ &= -\tfrac{1}{3}\kappa x, && \text{for} \quad x \in \mathscr{V}_{11} \oplus \mathscr{V}_{33}. \end{aligned}$$

Proof. In view of Lemma 4.1 it remains to show the above formula. For that, we take (2.9) and set there $w=v$, $x=u$, $z=y\in\mathscr{V}_1$. This yields

$$y\cdot(v\cdot v)=3(y\cdot v)\cdot v+v\cdot(y\cdot v)-(v\cdot y)\cdot v \qquad \text{for } y\in\mathscr{V}_1. \quad (4.15)$$

If $y\in\mathscr{V}_{13}$, then $v\cdot y=3y\cdot v$ by (4.13) and $v\cdot v=vvv=-9\kappa^{-1}u$ by (4.4). Now (4.15) implies

$$-9\kappa^{-1}y=\tfrac{1}{3}v\cdot(v\cdot y)=\tfrac{1}{3}L(v,v)^2y=3\kappa^{-2}L(u,u)^2y \quad \text{for} \quad y\in\mathscr{V}_{13}. \quad (4.16)$$

If $y\in\mathscr{V}_{11}$, then by definition of $\mathscr{V}_{-1}$ and Lemma 2.2 we have $v\cdot y=y\cdot v$, thus (4.15) reads as

$$-9\kappa^{-1}y=3v\cdot(v\cdot y)=27\kappa^{-2}L(u,u)^2y \quad \text{for} \quad y\in\mathscr{V}_{11} \qquad (4.17)$$

(note that $v\cdot y\in\mathscr{V}_{11}$ for $y\in\mathscr{V}_{11}$ by Lemma 4.1).

We take (2.9) again with $x=y=v$, $z=u$, and w replaced by y. This yields

$$\begin{aligned} y\cdot(v\cdot v)-3(v\cdot v)\cdot y &= 5(y\cdot v)\cdot v-v\cdot(y\cdot v) \\ &\quad -2v\cdot(v\cdot y)-4(v\cdot y)\cdot v. \qquad (4.18) \end{aligned}$$

As mentioned we have $v\cdot y=y\cdot v$ and $v\cdot y\in\mathscr{V}_{13}$ for $y\in\mathscr{V}_{31}$; $v\cdot y=-y\cdot v$ and $v\cdot y\in\mathscr{V}_{33}$ for $y\in\mathscr{V}_{33}$. Thus (4.18) implies (4.16) for $y\in\mathscr{V}_{31}$ and (4.17) for $y\in\mathscr{V}_{33}$, which proves the proposition.

Corollary 1. *For $x\in\mathscr{V}$* we have

$$\hat{\hat{x}}=-3\kappa^{-1}x.$$

Proof. Compare the formula in the proposition with (4.14) and (4.3).

Corollary 2. *Let β be an invariant bilinear form on $\mathscr{V}$. With the notations $\mathscr{U}=\mathscr{V}_{11}\oplus\mathscr{V}_{33}$* and *$\mathscr{W}=\mathscr{V}_{13}\oplus\mathscr{V}_{31}$ we have*

(a) $\beta(\mathscr{U},\mathscr{W})=\{0\}$,

(b) $\beta(\mathscr{V}_{ij},\mathscr{V}_{ij})=\{0\}$ *for $i,j=1,3$, $i\neq j$.*

In particular, if β is nondegenerate, then the restrictions of β to $\mathscr{U}\times\mathscr{U}$ and $\mathscr{W}\times\mathscr{W}$ are also nondegenerate and $\dim\mathscr{V}_{13}=\dim\mathscr{V}_{31}$.

Proof. For $x \in \mathscr{U}$, $y \in \mathscr{W}$ we have $\beta(x, y) = -3\kappa^{-1}\beta(L(u,u)^2 x, y) = -3\kappa^{-1}\beta(x, L(u,u)^2 y) = 9\beta(x, y)$, hence $\beta(x, y) = 0$. For $x, y \in \mathscr{V}_{13}$ we have $\beta(x, y) = \beta(xvu, y) = -\beta(x, yuv) = 3\beta(x, y)$, so $\beta(x, y) = 0$. Similar for $\mathscr{V}_{31}$.

5. A Binary Composition

Let $\mathscr{V}$ be a $\mathscr{T}$-ternary algebra with regular element u. As before we write $v = \kappa^{-1}u^3$, where κ is given by $L(u, u^3) = \kappa Id$. This section is devoted to the study of some properties of the binary composition

$$\begin{aligned} xy &:= xvy, && \text{for} \quad x \in \mathscr{V}_1, \\ &:= \tfrac{1}{2}\langle u, x\rangle \hat{y}, && \text{for} \quad x \in \mathscr{V}_3. \end{aligned}$$

Obviously we have

$$ux = xu, \qquad vx = \hat{x} \qquad (x \in \mathscr{V}), \tag{5.1}$$

$$v^2 = \tfrac{1}{3}vvv = -3\kappa^{-1}u; \tag{5.2}$$

the latter equality by (3.4).

LEMMA 5.1. *For all $x \in \mathscr{V}_3$ and all $y \in \mathscr{V}$ we have*

(a) $$xy = -xvy^s + 2y^s vx,$$

(b) $$\begin{aligned} (xy)\hat{} &= x\hat{y}, && \text{if} \quad x \in \mathscr{V}_{31}, \\ &= -x\hat{y}, && \text{if} \quad x \in \mathscr{V}_{33}. \end{aligned}$$

(*Recall that s is defined by $s(x) = 2r^{-1}(x) - x (x \in \mathscr{V})$, where $r(x) = xvu$.*)

Proof. For $x \in \mathscr{V}_3$ we have by definition $2xy = \langle u, x\rangle \hat{y} = -x(vy^s v)u + u(vy^s v)x = -y^s v(xvu) + xv(y^s vu) + (y^s vx)vu + y^s v(uvx) - uv(y^s vx) - (y^s vu)vx = -y^s vx^r + xvy^{sr} + (y^s vx)^r - y^{sr}vx$. Since $(y^s vx)^r = 2y^s vx - xvy^s + y^{sr}vx - xvy^{sr} - y^s vx^r + x^r vy^s$ by (2.3) and $x^r = 3x$, we have $2xy = 2xvy^s - 4y^s vx$, and (a) has been proved. To show formula (b) remember that $(x\hat{y}z)\hat{} = -\hat{x}y\hat{z}$ for all $x, y, z \in \mathscr{V}$ (Proposition 2.1). We thus find $2(xy)\hat{} = (\langle u, x\rangle \hat{y})\hat{} = -\langle v, \hat{x}\rangle y$.

Set $\xi_x := \kappa^{-1}$ for $x \in \mathscr{V}_{31}$ and $\xi_x := -3\kappa^{-1}$ for $x \in \mathscr{V}_{33}$. Then $\hat{x} = \xi_x uux$ by (4.14) and (4.3), hence $2(xy)\hat{} = -\xi_x\langle v, uux\rangle y = \xi_x\langle uuv, x\rangle y - \xi_x\langle u, \langle v, x\rangle u\rangle y = -3\xi_x\langle u, x\rangle y - \xi_x\langle u, xuv\rangle y + \xi_x\langle u, vux\rangle y = -3\xi_x\langle u, x\rangle y$ for $x \in \mathscr{V}_{31}$, and $= -\xi_x\langle u, x\rangle y$ for $x \in \mathscr{V}_{33}$. We conclude $2(xy)\hat{} = \mp 3\kappa^{-1}\langle u, x\rangle y = \pm\langle u, x\rangle \hat{\hat{y}} = \pm 2x\hat{y}$ for $x \in \mathscr{V}_{31}, \mathscr{V}_{33}$, respectively.

Proposition 5.1. *With respect to the above composition, the vector space $\mathscr{V}$ is an algebra with unit element u and involution*

$$x \mapsto x^\tau = 2x - xvu \qquad (x \in \mathscr{V}).$$

Occasionally we denote this algebra by $(\mathscr{V}, \tau)$.

Proof. The fact that u is the unit element is obvious and has already been stated in (5.1). Moreover, τ is of period two. Before we show that τ is an antiautomorphism we note that

$$xvy = xy + yx - yx^\tau \qquad (x, y \in \mathscr{V}), \tag{5.3}$$

and thus

$$xy^\tau - yx^\tau = -\langle x, y\rangle\, v \in \mathscr{V}_3 \qquad (x, y \in \mathscr{V}). \tag{5.4}$$

We set $\xi_1 := 1$, $\xi_3 = -\frac{1}{3}$. It is immediate from the definition of xy and Lemma 5.1(a), respectively, that $xy^\tau + yx^\tau = \xi_i(xvy + yvx)$ for $x, y \in \mathscr{V}_i$, whereas $xy^\tau + yx^\tau = -3xvy + yvx$ for $x \in \mathscr{V}_1, y \in \mathscr{V}_3$. We thus have $\xi_i^{-1}\langle u, xy^\tau + yx^\tau\rangle = \langle u, xvy\rangle + \langle u, yvx\rangle = -\langle xvu, y\rangle + \langle x, \langle u, y\rangle\, v\rangle - \langle yvu, x\rangle + \langle y, \langle u, x\rangle\, v\rangle = 0$ for all $x, y \in \mathscr{V}_i$, hence $xy^\tau + yx^\tau \in \mathscr{V}_1$. In the same way one shows that this is true for $x \in \mathscr{V}_1, y \in \mathscr{V}_3$. So we have proved

$$xy^\tau + yx^\tau \in \mathscr{V}_1 \qquad (x, y \in \mathscr{V}). \tag{5.5}$$

From (5.4) and (5.5) we conclude $2(xy^\tau)^\tau = ((xy^\tau - yx^\tau) + (xy^\tau + yx^\tau))^\tau = -xy^\tau + yx^\tau + xy^\tau + yx^\tau = 2yx^\tau$.

Proposition 5.2. *For all $x \in \mathscr{V}_3$ and all $y \in \mathscr{V}$ we have the weak alternative law*

$$(x, x, y) = (y, x, x) = 0,$$

where $(x, y, z) = (xy)\, z - x(yz)$ is the associator of x, y, z.

Proof. It suffices to show $(x, x, y) = 0$, for we have the involution τ. But this equation follows by a simple computation from formulas (5.6), (5.7) which we are going to prove. We adopt the notation $x \cdot y = xvy$ and $(x, y, z)_. = (x \cdot y) \cdot z - x \cdot (y \cdot z)$.

$$(x, x, y)_. = -2(x \cdot y) \cdot x - 4x \cdot (y \cdot x) + 8(y \cdot x) \cdot x \qquad (x \in \mathscr{V}_3, y \in \mathscr{V}_1), \tag{5.6}$$

$$(x, x, y)_. = -2(x \cdot y) \cdot x \qquad (x, y \in \mathscr{V}_3). \tag{5.7}$$

In order to verify these identities, we set $y = x \in \mathscr{V}_3$, $z = u$, and $w \to y$ in (2.9). Thus, for $y \in \mathscr{V}_1$,

$$\begin{aligned} y \cdot (x \cdot x) - (x \cdot x) \cdot y + 15(y \cdot x) \cdot x - 15x \cdot (y \cdot x) \\ + 6x \cdot (x \cdot y) - 6(x \cdot y) \cdot x = 0, \end{aligned} \tag{a}$$

and, for $y \in \mathscr{V}_3$,

$$\begin{aligned} y \cdot (x \cdot x) - 3(x \cdot x) \cdot y - 5(y \cdot x) \cdot x + x \cdot (y \cdot x) \\ + 2x \cdot (x \cdot y) - 4(x \cdot y) \cdot x = 0. \end{aligned} \tag{b}$$

We choose $x = u$, $z = y \in \mathscr{V}_1$, and $w \to x \in \mathscr{V}_3$ in (2.9) and get

$$y \cdot (x \cdot x) - 3(y \cdot x) \cdot x - x \cdot (y \cdot x) + (x \cdot y) \cdot x = 0. \tag{c}$$

Setting $w = x$ and $z = u$ in (2.9) we obtain

$$-2(y \cdot x) \cdot x + x \cdot (x \cdot y) - (x \cdot y) \cdot x = 0. \tag{d}$$

Finally, $y = x$, $z \to y$ in (V) yields

$$\begin{aligned} -y \cdot (x \cdot x) + (x \cdot x) \cdot y + (y \cdot x) \cdot x - x \cdot (y \cdot x) \\ +2x \cdot (x \cdot y) - 2(x \cdot y) \cdot x = 0. \end{aligned} \tag{e}$$

Now, substraction of (e) plus twice (d) from (b) gives (5.7), and (5.6) follows by subtraction of (a) minus twice (e) from (c).

Remark. In the proof of Theorem 5.1 we need

$$x \cdot (y \cdot z) - y \cdot (x \cdot z) = -(x \cdot y^s - y \cdot x^s) \cdot z \tag{5.8}$$

for $x \in \mathscr{V}_i, y \in \mathscr{V}_j, z \in \mathscr{V}$, $i \neq j$. This follows immediately from (2.9) by putting $x = u$, $w \to y \in \mathscr{V}_3$ and $y \to x \in \mathscr{V}_1$ for $i = 1$, $j = 3$ and clearly this proves the formula.

THEOREM 5.1. *For all x, y, z we have*

(a) $x\hat{y}z = (xy^\tau) z + (zy^\tau) x - (zx^\tau) y$,

(b) $2(xy^\tau) z = x\hat{y}z + y\hat{x}z - \langle x, y \rangle \hat{z}$.

Proof. Part (a) is an immediate consequence of (b), for the right-hand side of (a) is equal to $\frac{1}{2}(x\hat{y}z + y\hat{x}z - \langle x, y \rangle \hat{z} + z\hat{y}x + y\hat{z}x - \langle z, y \rangle \hat{x} - z\hat{x}y - x\hat{z}y + \langle z, x \rangle \hat{y}) = x\hat{y}z$.

(b) We assume first that $x, y \in \mathscr{V}_i$, $i = 1, 3$. In this case we get $x\hat{y}z + y\hat{x}z = (x^s \cdot y + y^s \cdot x) \cdot z$ from (2.8). The right-hand side is equal to $(xy + yx) \cdot z$ if $i = 1$, and equal to (cf. (5.3)) $-\frac{1}{3}(xy + 2yx + yx +$

$2xy) \cdot z = -(xy + yx) \cdot z$ if $i = 3$, hence in either case equal to $(xy^\tau + yx^\tau) \cdot z = (xy^\tau + yx^\tau) z$, which, by (5.4), proves (a) for $x, y \in \mathscr{V}_1$. For the rest it is enough to assume $x \in \mathscr{V}_1, y \in \mathscr{V}_3$. Now (2.8) yields $3(x\hat{y}z + y\hat{x}z) = -4x \cdot (y \cdot z) + 4y \cdot (x \cdot z) - (y \cdot x) \cdot z + 3(x \cdot y) \cdot z =$ $3(y \cdot x) \cdot z - 9(x \cdot y) \cdot z$, the latter because of (5.8). Applying (5.3) we arrive at formula (b).

LEMMA 5.2. *Suppose that $-3\kappa^{-1}$ is a square in K, say $-3\kappa^{-1} = \zeta^2$. Then*

(a) *$d := \frac{1}{2}(u + \zeta^{-1}v)$ is an idempotent in $\mathscr{V}$,*
(b) *$(d, x, d) = 0$,*
(c) *$(d, x, y) = -(x, d, y) = (x, y, d)$,*
(d) *$X(X-1)$ is the minimal polynomial of $L(d)$ and $R(d)$.*

Proof. Part (a) is obvious from (5.2); (b), (d) are immediate consequences of (c). Since we have the involution τ, only the left equation needs a proof and we can replace d by v. If $x \in \mathscr{V}_3$, this is a special case of Proposition 5.2. Let $x \in \mathscr{V}_1$. Then we have $(xvy)^\wedge = (x\hat{u}y)^\wedge = -\hat{x}u\hat{y} = -(vxv)\, u\hat{y} = v(xvu)\, \hat{y} = vx\hat{y} = xv\hat{y} + \langle v, x \rangle \hat{y}$ by Proposition 2.1, hence $v(xy) = x(vy) - (vx^\tau - xv^\tau) y = x(vy) - (vx) y - (xv) y$.

As for alternative algebras, Lemma 5.2 gives a Peirce decomposition of the algebra $(\mathscr{V}, \tau)$ with respect to the idempotent d. The usual proof goes through.

PROPOSITION 5.3. *Suppose $-3\kappa^{-1} = \zeta^2$ for suitable $\zeta \in K$. Let $d = \frac{1}{2}(u + \zeta^{-1}v)$ and*

$$\mathscr{V}^{ij} := \{x \in \mathscr{V};\ dx = ix,\ xd = jx\}, \qquad i, j = 0, 1.$$

Then we have

(a) $\mathscr{V} = \mathscr{V}^{11} \oplus \mathscr{V}^{10} \oplus \mathscr{V}^{01} \oplus \mathscr{V}^{00}$,
(b) $\mathscr{V}^{ij}\mathscr{V}^{ij} \subset \mathscr{V}^{ji}$, $\mathscr{V}^{ij}\mathscr{V}^{kl} \subset \delta_{jk}\mathscr{V}^{il}$, if $(i, j) \neq (k, l)$.

Remark. To see how the spaces $\mathscr{V}^{ij}$ are related to the spaces $\mathscr{V}_{ij}$, we recall that

$$\mathscr{U} = \mathscr{V}_{11} \oplus \mathscr{V}_{33}, \qquad \mathscr{W} = \mathscr{V}_{13} \oplus \mathscr{V}_{31},$$

are invariant under $x \mapsto \zeta uux (x \in \mathscr{V})$, and the square of this operator is the identity on $\mathscr{U}$, whereas it is equal to $9Id$ when restricted to $\mathscr{W}$. Hence $\mathscr{U}$ and $\mathscr{W}$ decompose as

$$\mathscr{U} = \mathscr{U}^+ \oplus \mathscr{U}^-, \qquad \mathscr{U}^\varepsilon = \{x \in \mathscr{V};\ \zeta uux = \varepsilon x\},$$

$$\mathscr{W} = \mathscr{W}^+ \oplus \mathscr{W}^-, \qquad \mathscr{W}^\varepsilon = \{x \in \mathscr{V};\ \zeta uux = 3\varepsilon x\}.$$

Now an easy computation gives

$$\mathscr{V}^{11} = \mathscr{W}^-, \qquad \mathscr{V}^{10} = \mathscr{U}^+, \qquad \mathscr{V}^{01} = \mathscr{U}^-, \qquad \mathscr{V}^{00} = \mathscr{W}^+.$$

Under the assumption of the proposition that $-3\kappa^{-1}$ be a square in K, Corollary 2 to Proposition 4.3 concerning invariant bilinear forms can be extended as follows

COROLLARY. *Let β be an invariant bilinear form on $\mathscr{V}$. Then $\beta(\mathscr{U}^\varepsilon, \mathscr{U}^\varepsilon) = \{0\}$, $\beta(W^\varepsilon, \mathscr{W}^\varepsilon) = \{0\}$ for $\varepsilon = \pm$. In particular, if β is nondegenerate, we have* $\dim \mathscr{U}^+ = \dim \mathscr{U}^-$, $\dim \mathscr{W}^+ = \dim \mathscr{W}^-$.

6. CLASSIFICATION

We have seen in Section 5 that each regular element in a $\mathscr{J}$-ternary algebra determines a binary composition $(x, y) \mapsto xy$ and an involution τ such that $x\hat{y}z = (xy^\tau)z + (zy^\tau)x - (zx^\tau)y$, where $\hat{y} = vy$ with $v = \kappa^{-1}uuu$. We shall show in this section that a reduced simple J-ternary algebra $\mathscr{V}$ of degree two is either isomorphic to one of type (II) or else isomorphic to one of type (I_n), $n \geqslant 2$, according to $\mathscr{V}_3$ generates $(\mathscr{V}, \tau)$ or not. This together with Proposition 4.1 and Theorem 5.1 completes the classification.

For the remaining part of this article we assume that the Jordan algebra $\mathscr{J}(\mathscr{V})$ corresponding to the $\mathscr{J}$-ternary algebra $\mathscr{V}$ is reduced of degree two and simple. In particular $\dim \mathscr{V} \geqslant 3$ and the defining symmetric bilinear form μ of $\mathscr{J}(\mathscr{V})$ is nondegenerate.

We adopt the notations and conventions of the foregoing sections. In particular, u denotes a fixed regular element, $L(u, u^3) = \kappa Id$, $v^3 = -9\kappa^{-1}u$, and $\mathscr{V} = \mathscr{V}_{11} \oplus \mathscr{V}_{13} \oplus \mathscr{V}_{31} \oplus \mathscr{V}_{33}$ the Peirce decomposition of $\mathscr{V}$ with respect to u. We have introduced (Section 3) a linear form t and a symmetric bilinear form $(-,-)$ on $\mathscr{V}_3$ such that the Jordan algebra $(\mathscr{V}_3, \circ)$, which is isomorphic to $\mathscr{J}(\mathscr{V})$ (see Proposition 3.1), satisfies

$$x \circ y = t(x)y + t(y)x - (x, y)v. \tag{6.1}$$

Recall that

$$t(x) = \beta(u, x) \tag{6.2}$$

and

$$(x, y) = 2\beta(u, x)\beta(u, y) - \tfrac{1}{2}\beta(\langle u, x\rangle u, y) \qquad \text{for} \quad x, y \in \mathscr{V}_3. \tag{6.3}$$

Since $x = -\frac{1}{3}\kappa\hat{\hat{x}}$ (Corollary 1 to Proposition 4.3), we conclude from Proposition 3.1 that

$$x \circ y = -\tfrac{1}{6}\kappa(x\hat{y} + y\hat{x}), \tag{6.4}$$

in particular

$$x \circ x = -\tfrac{1}{3}\kappa x\hat{x}. \tag{6.5}$$

Combining this with formula (6.1) we get

$$x\hat{x} = -3\kappa^{-1}(2t(x)\,x - (x, x)\,v). \tag{6.6}$$

We define subspaces

$$\mathscr{C}_1 = Ku \oplus \mathscr{C}_1^0, \qquad \mathscr{C}_1^0 = \{x \in \mathscr{V}_{31}\,;\beta(u, x) = 0\},$$
$$\mathscr{C}_2 = Ku \oplus \mathscr{C}_2^0, \qquad \mathscr{C}_2^0 = V_{33} \oplus Kv,$$

and claim that these are composition algebras with respect to the product of $\mathscr{V}$ introduced in Section 5. As the main step in that direction we prove

LEMMA 6.1. *For all* $x \in \mathscr{C}_i$ we have

$$x^2 = 2\sigma_i(x)\,x - n_i(x)\,u, \tag{6.7}$$

where

$$\begin{aligned} &\sigma_i(\xi u + x_0) := \xi && (\xi \in K, x_0 \in \mathscr{C}_i^0),\\ &n_1(\xi u + x_0) := \xi^2 - 3\kappa^{-1}(x_0, x_0) && (\xi \in K, x_0 \in \mathscr{C}_1^0),\\ &n_2(\xi u + x_0) := \xi^2 + 3\kappa^{-1}(x_0, x_0) && (\xi \in K, x_0 \in \mathscr{C}_2^0). \end{aligned}$$

Proof. First suppose that $x \in \mathscr{C}_1^0$. Then, by Lemma 5.1, $(x\hat{x})\hat{} = x\hat{\hat{x}} = -3\kappa^{-1}x^2$. Now (6.6) yields $3\kappa^{-1}(x, x)\,\hat{v} = -3\kappa^{-1}x^2$, thus $x^2 = -(x, x)\,\hat{\hat{u}} = 3\kappa^{-1}(x, x)\,u = -n_1(x)$. Using this result, an easy computation gives the desired formula for arbitrary $x = \sigma_1(x) + x_0, x_0 \in \mathscr{C}_1^0$.

Next suppose that $x \in \mathscr{V}_{33}$. Again by Lemma 5.1 we have $(x\hat{x})\hat{} = -x\hat{\hat{x}} = 3\kappa^{-1}x^2$, and (6.6) yields $3\kappa^{-1}x^2 = 3\kappa^{-1}(x, x)\,\hat{v}$, hence $x^2 = (x, x)\,\hat{\hat{u}} = -3\kappa^{-1}(x, x)\,u = -n_2(x)\,u$. If x is an arbitrary element of $\mathscr{C}_2^0$, it can be written as $x = x_3 + \gamma v$, where $x_3 \in \mathscr{V}_{33}$ and $\gamma \in K$. We thus have $x^2 = x_3^2 + \gamma(x_3 v + v x_3) + \gamma^2 v^2 = -n_2(x_3)\,u - 3\kappa^{-1}\gamma^2 u$, since $x_3 v + v x_3 = x_3\hat{u} + \hat{x}_3 = -(x_3 u)\hat{} + \hat{x}_3 = 0$. On the other hand, $(x, x) = (x_3 + \gamma v, x_3 + \gamma v) = (x_3, x_3) + 2\gamma(v, x_3) + \gamma^2(v, v) = (x_3, x_3) + \gamma^2$ by (6.3) and (3.5). Hence $3\kappa^{-1}(x, x)\,u = 3\kappa^{-1}(x_3, x_3)\,u + 3\kappa^{-1}\gamma^2 u = -x_3^2 - \gamma^2 v^2 = -x^2$. It is now easy to check the assertion for the general case, i.e., for all those $x \in \mathscr{C}_2$ for which $\sigma_2(x) \neq 0$.

Linearisation of (6.7) yields

$$\tfrac{1}{2}(xy+yx)=\sigma_i(x)\,y+\sigma_i(y)\,x-n_i(x,y)\,u, \tag{6.8}$$

where

$$n_i(x,y)=\tfrac{1}{2}(n_i(x+y)-n_i(x)-n_i(y))$$

for all $x,y\in\mathscr{C}_i$. By an easy computation we get

$$n_i(\xi u+x_0,\zeta u+y_0)=\xi\zeta+(-1)^i 3\kappa^{-1}(x_0,y_0) \tag{6.9}$$

for all ξ, $\zeta\in K$ and all $x_0,y_0\in\mathscr{C}_i^0$. Since we have supposed that $\mathscr{T}(\mathscr{V})$ (i.e., μ) is nondegenerate, so is $(-,-)$ and hence n_i, $i=1,2$.

PROPOSITION 6.1. *$\mathscr{C}_1$ and $\mathscr{C}_2$ are composition algebras with canonical involutions $\tau_i=\tau|\mathscr{C}_i$, $i=1,2$.*

Proof. Since $u^\tau=u$ and $x^\tau=-x$ for all $x\in\mathscr{C}_1^0\cup\mathscr{C}_2^0\subset\mathscr{V}_3$, it is obvious from the definition of σ_i, that

$$x^\tau=2\sigma_i(x)\,u-x \tag{6.10}$$

holds for all $x\in\mathscr{C}_i$. An immediate consequence of (6.10) and Lemma 6.1 is

$$xx^\tau=x^\tau x=n_i(x)\,u \qquad (x\in\mathscr{C}_i). \tag{6.11}$$

We have already pointed out that n_i is nondegenerate ($i=1,2$) and by Proposition 5.2 we know that the alternative law is valid in $\mathscr{C}_1$ and $\mathscr{C}_2$. Now by [6, Definition 3, p. 162] the proposition is proved when we have shown that $\mathscr{C}_1$ and $\mathscr{C}_2$ are closed under multiplication. In order to show this we observe first that $xy+yx\in Ku\oplus\mathscr{V}_3$ for $x,y\in\mathscr{C}_i^0$ by (6.8), and $xy-yx=\langle x,y\rangle v\in\mathscr{V}_3$ by (5.4). Hence $xy\in Ku\oplus\mathscr{V}_3$ for all $x,y\in\mathscr{C}_i$. Without loss of generality we can assume that K is algebraically closed. Thus we can apply Proposition 5.3. We write $\mathscr{C}_2^{ij}:=\mathscr{V}^{ij}\cap\mathscr{C}_2$, $i,j=0,1$, and get

$$\mathscr{C}_2=\mathscr{C}_2^{11}\oplus\mathscr{C}_2^{10}\oplus\mathscr{C}_2^{01}\oplus\mathscr{C}_2^{00}. \tag{6.12}$$

Proposition 5.3(b) and the fact that $xy\in Ku\oplus\mathscr{V}_3$ for $x,y\in\mathscr{C}_2^0$, imply that $\mathscr{C}_2$ is closed under multiplication and that (6.12) is actually the Peirce decomposition of the composition algebra $\mathscr{C}_2$. If $x,y\in\mathscr{C}_1^0$, then $x,y\in\mathscr{W}$, hence $x,y\in\mathscr{W}\cap(Ku\oplus\mathscr{V}_3)=Ku\oplus\mathscr{V}_{31}$. Furthermore, $\beta(u,xy)=\tfrac{1}{2}\beta(u,\langle u,x\rangle\hat{y})=\tfrac{1}{2}\beta(\langle u,x\rangle u,\hat{y})$. Since $\langle u,x\rangle u$ and $\hat{y}$ are both in $\mathscr{V}_{13}$ by Lemma 4.1 and $\beta(\mathscr{V}_{13},\mathscr{V}_{13})=\{0\}$ by Corollary 2 to Proposition 4.3 we see that $\beta(u,xy)=0$, hence $xy\in\mathscr{C}_1$, which finishes the proof.

For the rest of the paper we denote by $\mathscr{X}$ the subalgebra of $(\mathscr{V}, \tau)$ which is generated by $\mathscr{C}_1 \cup \mathscr{C}_2$.

PROPOSITION 6.2. *The subalgebra* $\mathscr{X}$ *of* $(\mathscr{V}, \tau)$ *is invariant under* τ. *Moreover,* $(\mathscr{X}, \tau)$ *and* $(\mathscr{C}_1 \otimes \mathscr{C}_2, \tau_1 \otimes \tau_2)$ *are isomorphic as algebras with involutions.*

Since $\mathscr{C}_1$ and $\mathscr{C}_2$ are invariant under τ (Proposition 6.1), the same holds for $\mathscr{C}_1 \cup \mathscr{C}_2$, hence for $\mathscr{X}$. In order to establish an isomorphism, we first prove

LEMMA 6.2. *For* $x_i \in \mathscr{C}_i$ *and* $z \in \mathscr{V}$ *we have*

(a) $x_1 x_2 = x_2 x_1$,

(b) $(x_1 x_2) z = x_1(x_2 z) = x_2(x_1 z)$.

Proof. (a) We can assume that $x_i \in \mathscr{C}_i^0$. If $x_2 = v$, we have $x_1 x_2 - x_2 x_1 = x_1 v - v x_1 = x_1 \hat{u} - \hat{x}_1$ (5.1). Since $x_1 \in \mathscr{V}_{31}$, Lemma 5.1(b) implies that $x_1 \hat{u} = \hat{x}_1$, hence $x_1 x_2 = x_2 x_1$, if $x_2 = v$. Suppose now that $t(x_2) = 0$, i.e., $x_2 \in \mathscr{V}_{33}$. Since $t(x_1) = 0$ for all $x_1 \in \mathscr{C}_1^0$, Lemma 5.1(*b*), (6.2)–(6.4) yield $(x_1 x_2 - x_2 x_1)\hat{} = x_1 \hat{x}_2 - x_2 \hat{x}_1 = -6\kappa^{-1} x_1 \circ x_2 = 6\kappa^{-1}(x_1, x_2)\, v = -3\kappa^{-1}\beta(\langle u, x_1\rangle\, u, x_2)$. But $\langle u, x_1\rangle\, u \in \mathscr{V}_{13}$ by Lemma 4.1, so $x_2 \in \mathscr{V}_{33}$ and Corollary 2 to Proposition 4.3 forces $x_1 x_2 - x_2 x_1$ to be zero.

In order to prove (b), assume again $x_i \in \mathscr{C}_i^0$. Since $x_1 \in \mathscr{V}_{31}$, $x_2 \in \mathscr{V}_{33}$, Lemma 5.1(b) yields $(x_1 z)\hat{} = x_1 \hat{z}$, $(x_2 z)\hat{} = -x_2 \hat{z}$, hence

$$\begin{aligned}
4(x_1(x_2 z) - x_2(x_1 z)) &= 2(-\langle u, x_1\rangle(x_2 \hat{z}) - \langle u, x_2\rangle(x_1 \hat{z})) \\
&= 3\kappa^{-1}(\langle u, x_1\rangle(\langle u, x_2\rangle\, z) + \langle u, x_2\rangle(\langle u, x_1\rangle\, z)) \\
&= 6\kappa^{-1}(\langle u, x_1\rangle\langle u, x_2\rangle)\, z = 24\kappa^{-1}(\eta^{-1}(x_1)\, \eta^{-1}(x_2))\, z \\
&= 24\kappa^{-1}(x_1 \circ x_2)\, \hat{z} = -4(x_1 \hat{x}_2 + x_2 \hat{x}_1)\, \hat{z} = -4(x_1 x_2 - x_2 x_1)\hat{}\, \hat{z}.
\end{aligned}$$

Since we have already proved that $x_1 x_2 = x_2 x_1$, the first equality of (b) has been proved. The second one follows from the first one by means of the linearised version of the weak alternative law (Proposition 5.2) which says that $(x_1 x_2)\, z + (x_2 x_1)\, z = x_1(x_2 z) + x_2(x_1 z)$, hence $(x_2 x_1)\, z = x_1(x_2 z)$, and application of (a) finishes the proof of Lemma 6.2.

Now, by the universal property of the tensor product, there exists a unique linear mapping Φ such that

$$\Phi\colon\ C_1 \otimes C_2 \to V, \qquad \Phi(x_1 \otimes x_2) = x_1 x_2 \qquad (x_i \in \mathscr{C}_i).$$

Since $\Phi \circ (\tau_1 \otimes \tau_2)(x_1 \otimes x_2) = x_1^{\tau_1} x_2^{\tau_2} = (x_2 x_1)^{\tau}$, we see that $\Phi \circ (\tau_1 \otimes \tau_2) =$

$\tau|_X \circ \Phi$. Next we show that Φ is an algebra homomorphism, using Lemma 6.2

$$\Phi(x_1 \otimes x_2)\, \Phi(x_1' \otimes x_2') = (x_1 x_2)(x_1' x_2') = x_1(x_2(x_1' x_2'))$$
$$= x_1(x_1'(x_2 x_2')).$$

Applying τ to the equality $(x_1 x_2)\, z = x_2(x_1 z)$, we see that $z(x_1 x_2) = (z x_1)\, x_2$ for all $z \in \mathscr{V}$, thus

$$x_1(x_1'(x_2 x_2')) = (x_1 x_1')(x_2 x_2') = \Phi((x_1 \otimes x_2)(x_1' \otimes x_2')).$$

Φ is injective: since the kernel of Φ is invariant under τ, it is an ideal of the simple ternary algebra $\mathscr{C}_1 \otimes \mathscr{C}_2$. But Φ is not identically zero, so $\ker \Phi = \{0\}$.

Now we formulate the classification theorem (part (a) of which has been proved by Proposition 6.2 and Theorem 5.1).

THEOREM 6.1. *Let $\mathscr{V}$ be a simple reduced $\mathscr{T}$-ternary algebra of degree two. Then $\mathscr{V}$ is isomorphic to one of type* (I_n), $n \geqslant 1$ *or to one of type* (II) *(see the examples in Section* 1 *for definition). More precisely, let* $(\mathscr{V}, \tau)$ *be the algebra with involution defined in Proposition* 5.1 *(with respect to an arbitrary regular element of $\mathscr{V}$) and let $\mathscr{X}$ be the subalgebra of* $(\mathscr{V}, \tau)$ *which is generated by $\mathscr{C}_1 \cup \mathscr{C}_2$ (see Proposition* 6.1).

(a) *If $\mathscr{X} = \mathscr{V}$, then $\mathscr{V}$ is of type* (I_1) *or of type* (II) *according to $\mathscr{C}_1$ and $\mathscr{C}_2$ are both associative or not.*

(b) *If $\mathscr{X} \neq \mathscr{V}$, then $\mathscr{C}_1$ and $\mathscr{C}_2$ (hence $\mathscr{X}$) are associative and $\mathscr{V}$ is of type* (I_n) *with suitable* $n \geqslant 2$.

Proof. (b) By Theorem 5.1(a) and Proposition 6.2, $\mathscr{X}$ is a simple $\mathscr{T}$-ternary subalgebra of $\mathscr{V}$. In particular, $\langle -, - \rangle$ is nondegenerate on $\mathscr{X} \times \mathscr{X}$ [5, Theorem 6.1]. Therefore, the skew-symmetric nondegenerate invariant bilinear form

$$\beta(x, y) := \lambda(\langle x, y \rangle) \qquad (x, y \in \mathscr{V})$$

(see Section 3 for the definition of λ) is nondegenerate when restricted to $\mathscr{X} \times \mathscr{X}$. This implies

$$\mathscr{V} = \mathscr{X} \oplus \mathscr{Y},$$

where $\mathscr{Y}$ is the orthogonal complement of $\mathscr{X}$ with respect to β.

LEMMA 6.3. *$\mathscr{Y}$ is either zero or a $\mathscr{T}$-ternary subalgebra of $\mathscr{V}$ with $\mathscr{T}(\mathscr{Y}) = \mathscr{T}(\mathscr{V})$. Moreover, $\langle \mathscr{X}, \mathscr{Y} \rangle = \{0\}$, $\mathscr{Y} \subset \mathscr{V}_{11}$ and $\mathscr{X}\mathscr{Y} \subset \mathscr{Y}$, $\mathscr{Y}\mathscr{X} \subset \mathscr{Y}$, $\mathscr{Y}\mathscr{Y} \subset \mathscr{X}$.*

Proof. Let $\mathscr{Y} \neq \{0\}$. Then $\mathscr{Y}$ is $\mathscr{T}$ invariant because $\mathscr{X}$ and β are $\mathscr{T}$ invariant. Thus, by Proposition 1.1, $\mathscr{Y}$ is a $\mathscr{T}$-ternary subalgebra of $\mathscr{V}$. It follows from formula (1.2), that $\langle \mathscr{Y}, \mathscr{Y} \rangle$ is an ideal of $\mathscr{T}(\mathscr{V})$, hence $\langle \mathscr{Y}, \mathscr{Y} \rangle$ is zero or equal to $\mathscr{T}(\mathscr{V})$. But $\langle \mathscr{Y}, \mathscr{Y} \rangle = \{0\}$ implies $\langle \mathscr{Y}, \mathscr{V} \rangle = \{0\}$, hence $\mathscr{Y} = \{0\}$, contrary to our assumption. It was shown in the proof of [5, Theorem 4.1], that $\beta(ax, y) = \lambda(a\langle x, y \rangle)$ for all $a \in \mathscr{T}(\mathscr{V})$, and all $x \in \mathscr{X}, y \in \mathscr{Y}$. But the left-hand side is zero, which forces $\langle x, y \rangle = 0$ (since λ is nondegenerate), thus $\langle \mathscr{X}, \mathscr{Y} \rangle = \{0\}$. To show $\mathscr{Y} \subset \mathscr{V}_{11}$, we recall that $\mathscr{V}_{31} \subset \mathscr{V}_3 \subset \mathscr{X}$. We have $v\mathscr{X} \subset \mathscr{X}$ (since $v \in \mathscr{X}$), and so $\mathscr{V}_{13} = v\mathscr{V}_{31} \subset \mathscr{X}$ (Proposition 4.3). By Corollary 2 to Proposition 4.3, β is nondegenerate on $\mathscr{V}_3 \oplus \mathscr{V}_{13}$, thus $\beta(x, y) = 0$ for all $x \in \mathscr{X}$ implies $y \in \mathscr{V}_{11}$. For $y \in \mathscr{Y}$, $x_1 \in \mathscr{X}$ we have $yx_1 = yvx_1$, since $y \in \mathscr{V}_1$, hence $\beta(yx_1, x_2) = \beta(yvx_1, x_2) = -\beta(y, x_2 x_1 v) = 0$ for all $x_2 \in \mathscr{X}$, proving $\mathscr{Y}\mathscr{X} \subset \mathscr{Y}$. Now, $\langle \mathscr{X}, \mathscr{Y} \rangle = \{0\}$ implies $\mathscr{X}\mathscr{Y} \subset \mathscr{Y}$. For $y_i \in \mathscr{Y}$ we have $\beta(y_1 y_2, y_3) = \beta(y_1 v y_2, y_3) = -\beta(y_1, y_3, y_2 v) = \beta(y_2 y_3 y_1, v)$. Since $\mathscr{Y}$ is a ternary subalgebra of $\mathscr{V}$ we conclude $\beta(y_1 y_2, y_3) = 0$ for all $y_i \in \mathscr{Y}$, hence $\mathscr{Y}\mathscr{Y} \subset \mathscr{X}$.

LEMMA 6.4. *For $x \in \mathscr{X}$, $y \in \mathscr{Y}$, $z \in \mathscr{V}$ we have*

(a) $xy = yx^\tau$,

(b) $x(yz) = y(x^\tau z)$, $(zy)x = (zx^\tau)y$.

Proof. Condition (a) is an immediate consequence of (5.4) and $\langle x, y \rangle = 0$ (Lemma 6.3). (b) First we assume $x \in \mathscr{V}_3$. By (2.2) and (2.5) we have $(yz)\hat{} = (yvz)\hat{} = -\hat{y}u\hat{z} = -vy\hat{z}$. Hence $2x(yz) + 2y(xz) = \langle u, x \rangle (yz)\hat{} + y(\langle u, x \rangle \hat{z}) = -\langle u, x \rangle (vy\hat{z}) + yv(\langle u, x \rangle \hat{z})$. We write $a = \langle u, x \rangle$, for short. We know that $-(av)y\hat{z} + y(av)\hat{z} = \langle y, av \rangle \hat{z} = 0$ and $v(ay)\hat{z} - (ay)v\hat{z} = \langle v, ay \rangle \hat{z} = 0$, since $\langle \mathscr{X}, \mathscr{Y} \rangle = \{0\}$ and $av \in \mathscr{X}$, $ay \in \mathscr{Y}$. It follows now from (III) that

$$\begin{aligned} 2x(yz) + 2y(xz) &= -(av)y\hat{z} + v(ay)\hat{z} - vy(a\hat{z}) \\ &+ a(yv\hat{z}) - (ay)v\hat{z} + y(av)\hat{z} = -vy(a\hat{z}) + a(yv\hat{z}) \\ &= -yv(a\hat{z}) + a(vy\hat{z}) = -2y(xz) - 2x(yz). \end{aligned}$$

Therefore $x(yz) = -y(xz) = y(x^\tau z)$. For arbitrary $x \in \mathscr{X}$ it suffices to consider $x = x_1 x_2$, where $x_i \in \mathscr{C}_i$. Moreover, we can assume that $x_i \in \mathscr{V}_3$. It follows from Lemma 6.2(b) that $(x_1 x_2)(yz) = x_1(x_2(yz)) = -x_1(y(x_2 z)) = y(x_1(x_2 z)) = y((x_1 x_2) z)$. Since x_1, x_2 commute and $x_i^\tau = -x_i$, this proves the lemma.

We are now in the position that we can prove the associativity of $\mathscr{C}_1$ and $\mathscr{C}_2$ (in case, $\mathscr{Y} \neq \{0\}$) as stated in Theorem 6.1. For this we can assume that the ground field is algebraically closed. Then we can find idempotents d_1, d_2 in $\mathscr{C}_i$ ($i=1$ or 2) such that $d_1+d_2=u$ (the unit element of $\mathscr{C}_i$) and $d_2^\tau=d_1$. Let $\mathscr{C}_i=Kd_1\oplus\mathscr{C}_i^{10}\oplus\mathscr{C}_i^{01}\oplus Kd_2$ be the corresponding Peirce decomposition of $\mathscr{C}_i$. If $\mathscr{C}_i$ were not associative, there would exist $x_1, x_2, x_3\in\mathscr{C}_i^{10}$ such that $(x_1,x_2,x_3)\neq 0$. Since $(x_1x_2)x_3\in Kd_2$ and $x_1(x_2x_3)\in Kd_1$, there exist scalars ξ,ζ, not both equal to zero, such that $(x_1,x_2,x_3)=\xi d_1+\zeta d_2=\xi d_1+\zeta d_1^\tau$. But $(x_1,x_2,x_3)^\tau=-(x_3^\tau,x_2^\tau,x_1^\tau)=(x_3,x_2,x_1)=-(x_1,x_2,x_3)$, so $\xi d_1+\zeta d_1^\tau=(-\xi d_1-\zeta d_1^\tau)^\tau=-\zeta d_1-\xi d_1^\tau$, hence $\xi=-\zeta$, which implies $\xi\neq 0$ and $(x_1,x_2,x_3)=\xi(d_1-d_2)=\xi(2d_1-u)$. Therefore,

$$(x_1,x_2,x_3)\,y=\xi(2d_1y-y),\qquad \xi\neq 0,\quad y\in\mathscr{Y}. \tag{$*$}$$

We compute the left-hand side. From Lemmas 6.3 and 6.4 we conclude that $(x_1(x_2x_3))\,y=(x_1y)(x_2x_3)^\tau=(x_2x_3)(x_1y)=(x_2(x_1y))\,x_3^\tau=(x_2(yx_1^\tau))\,x_3^\tau=(y(x_2^\tau x_1^\tau))\,x_3^\tau=x_3(y(x_2^\tau x_1^\tau))=y(x_3^\tau(x_2^\tau x_1^\tau))=((x_1x_2)\,x_3)\,y$, hence $(x_1,x_2,x_3)\,y=0$. From $(*)$ we get $2d_1y=y$. Multiplication with d_1 from the left yields $2d_1y=d_1y$, hence $d_1y=0$ and $0=\xi(2d_1y-y)=-\xi y$ which forces $y=0$, contrary to our assumption $\mathscr{Y}\neq\{0\}$. Thus $(x_1,x_2,x_3)\neq 0$ is impossible in this case, which proves that $\mathscr{C}_i$ is associative for $i=1,2$.

We are left with the problem of establishing an isomorphism from $\mathscr{V}$ to a $\mathscr{T}$-ternary algebra of type (I_n), $n\geqslant 1$ under the assumption $\mathscr{Y}\neq\{0\}$. Since $\mathscr{Y}$ is itself a simple reduced $\mathscr{T}$-ternary algebra of degree two (Lemma 6.3 and [5, Theorem 6.1]), we get (by induction on the dimension of $\mathscr{V}$) a decomposition

$$\mathscr{V}=\mathscr{X}_1\oplus\cdots\oplus\mathscr{X}_n$$

(direct vector space sum) with $\mathscr{X}_1=\mathscr{X}$, where each $\mathscr{X}_i$ is a $\mathscr{T}$-ternary algebra of type (I_1) with $\mathscr{T}(\mathscr{X}_i)=\mathscr{T}(\mathscr{V})$, $\langle\mathscr{X}_i,\mathscr{X}_j\rangle=\{0\}$ for $i\neq j$. After Proposition 4.1, each $\mathscr{X}_i$ contains a regular element u_i of $\mathscr{V}$. We fix one for each $i\geqslant 2$ and choose $v_i\in\mathscr{X}_i$ such that $L(u_i,v_i)=Id$. We prove

LEMMA 6.5. *For $2\leqslant i\leqslant n$ the linear mapping*

$$f_i:\ \mathscr{X}\to\mathscr{X}_i,\qquad f_i(x):=u_ix$$

is bijective and satisfies

(a) $f_i(x_1x_2)=x_1^\tau f_i(x_2)$

(b) $f_i(x_1)f_i(x_2)=x_2(u_i^2x_1^\tau)$

for $x_i\in\mathscr{X}$.

Proof. Since $L(u_i,v_i)=Id$, we have $v=u_iv_iv=v_ivu_i+2\langle v,u_i\rangle v_i=v_ivu_i=v_iu_i$ $(i\geqslant 2)$. Therefore, $u_ix=0$ implies $(u_ix)v_i=0$, hence

$(u_i v_i) x^\tau = 0$ or $v x^\tau = 0$ and consequently $x = 0$, proving that f_i is injective. In particular, $\dim \mathscr{X} \leqslant \dim \mathscr{X}_i$. We show that $g_i: \mathscr{X}_i \to \mathscr{X}$, $g_i(x_i) := u_i x_i$ $(i \geqslant 2)$ is injective too, which implies $\dim \mathscr{X}_i \leqslant \dim \mathscr{X}$, hence $\dim \mathscr{X} = \dim \mathscr{X}_i$, proving the surjectivity of f_i. Since $\mathscr{X}_i$ $(i \geqslant 2)$ is contained in $\mathscr{V}_1$, it follows immediately from Theorem 5.1(a) that $\ker g_i$ is an ideal of the simple $\mathscr{T}$-ternary algebra X_i. But $\ker g_i = \mathscr{X}_i$ would imply $u_i v_i = 0$, contrary to $u_i v_i = v_i u_i = v$ (see above).

Condition (a) is obvious from Lemma 6.4(b), and the same lemma gives $f_i(x_1) f_i(x_2) = (u_i x_1)(u_i x_2) = (u_i(u_i x_2)) x_1^\tau = (u_i(x_2^\tau u_i)) x_1^\tau = (x_2 u_i^2) x_1^\tau = x_2(u_i^2 x_1^\tau)$. Using Theorem 5.1(a) it is now easy to check that

$$f: \quad (x_1, \ldots, x_n) \mapsto x_1 \oplus f_2(x_2) \oplus \cdots \oplus f_n(x_n)$$

defines an isomorphism of a $\mathscr{T}$-ternary algebra of type (I_n) to $\mathscr{V}$ (with $d_i = u_i^2$).

References

1. B. N. Allison, A construction of Lie algebras from J-ternary algebras, *Amer. J. Math.* **98** (1976), 285–294.
2. B. N. Allison, A class of nonassociative algebras with involution, *Math. Ann.* **237** (1978), 133 – 156.
3. H. Braun and M. Koecher, "Jordan - Algebren," Springer-Verlag, Berlin/Heidelberg/ New York, 1966.
4. W. Hein, A construction of Lie algebras by triple systems, *Trans. Amer. Math. Soc.* **205** (1975), 79–95.
5. W. Hein, On the structure of reduced J-ternary algebras of degree >2, *Math. Z.* **176** (1981), 521–539.
6. N. Jacobson, "Structure and Representations of Jordan Algebras," Amer. Math. Soc. Coll. Publ. No. 39, Providence, R.I.
7. I. L. Kantor, Some generalisations of Jordan algebras, *Trudy Sem. Vektor. Tenzor. Anal.* **16** (1972), 407–499. [Russian]
8. I. L. Kantor, Models of exceptional Lie algebras, *Soviet Math. Dokl.* **14** (1973), 254–258.
9. M. Koecher, Imbedding of Jordan algebras into Lie algebras I, *Amer. J. Math.* **89** (1967), 787–815.